ON ORLICZ SEQUENCE SPACES III

BY

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ABSTRACT

It is proved that the set of p's such that l_p is isomorphic to a subspace of a given Orlicz space l_F forms an interval. Some examples and properties of minimal Orlicz sequence spaces are presented. It is proved that an Orlicz function space (different from l_2) is not isomorphic to a subspace of an Orlicz sequence space. Finally it is shown (under a certain restriction) that if two Orlicz function spaces are isomorphic, then they are identical (i.e. consist of the same functions).

1. Introduction

As the title of the paper indicates, this is a continuation of two previous papers ([8], [9]). However, apart from references to some results in the previous papers, this paper is quite self-contained.

In Section 2 we consider the set of p's for which l_p is isomorphic to a subspace of an Orlicz space l_F. We show that this set constitutes a closed interval (which may reduce to a single point). This interval is identical to the interval associated with an Orlicz space in various places in the literature. As a consequence we get that l_p is isomorphic to a subspace of a reflexive Orlicz space l_F if and only if it is isomorphic to a quotient space of l_F. (In general l_p need not, however, be isomorphic to a complemented subspace of l_F, as examples given in [9] and Section 3 below show.) This result exhibits a special property of l_p spaces: simple examples (given in [9]) show that an Orlicz space l_G may be isomorphic to a subspace of a reflexive Orlicz space l_F without l_G being a quotient space of l_F. As an easy application of this result concerning l_p subspaces of Orlicz spaces, we show that a well-known sufficient condition for every operator from l_F to l_G to be compact is also a necessary condition.

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Section 3 is devoted mainly to the study of minimal Orlicz sequence spaces. Minimal Orlicz spaces were introduced in [9] (their definition is given at the end of this introduction). As far as complemented subspaces are concerned, these spaces resemble the l_p spaces. It might even be true that like the l_p spaces, the minimal Orlicz sequence spaces are prime spaces, i.e. that every complemented subspace of such an X is either isomorphic to X itself or it is finite-dimensional. The first result in Section 3 is a characterization of the l_p spaces among the minimal Orlicz sequence spaces: a minimal Orlicz sequence space is isomorphic to an l_p space if and only if it has a unique symmetric basis, up to equivalence.

Section 3 also describes a way of representing an (essentially) general Orlicz function in a convenient form. Using this representation we give an example of a minimal Orlicz sequence space whose interval is non degenerate and an example of a minimal Orlicz sequence space which is not isomorphic to l_p but whose interval consists of a single point.

A remark should be made concerning the general nature of Sections 2 and 3. The results quoted above are in the general spirit of Banach space theory. The proofs given here do not, however, involve the investigation of Banach spaces. In [8] and [9] several Banach space theoretic properties of l_F were translated into properties of the flow T_t defined by $T_tF(x) = F(tx)/F(t)$. Sections 2 and 3 are mainly concerned with a direct investigation of the properties of this flow. In many places the argument resembles elementary and standard reasonings in topological dynamics. We do not make explicit use of results from topological dynamics since our setting is slightly different from the usual one (mainly because we identify equivalent Orlicz functions).

Another general remark concerning Sections 2 and 3 is this: we assume throughout that the function F generating the flow is convex. In most of the arguments, the convexity of F does not play any role (of course, in studying nonconvex F we have to allow also exponents p with $0 < p < 1$). Since for nonconvex F the sequence space l_F is not a Banach space, we have not pursued this possible generalization of the results of Sections 2 and 3.

The last section of this paper, Section 4, contains some results on Orlicz function spaces L_F and their relation to Orlicz sequence spaces. All the Orlicz function spaces we consider here are on the unit interval $[0, 1]$ endowed with the usual Lebesgue measure on it. The structure of Orlicz function spaces is naturally far more complicated than that of Orlicz sequence spaces. Some very interesting results on Orlicz function spaces were proved by probabilistic methods by Bretagnolle and Dacunha-Castelle [1]. Our results complement some points in their work.

The first result we prove in Section 4 is that unless L_F is a Hilbert space (i.e. $F(x)$ is equivalent at ∞ to x^2) the space L_F cannot be embedded isomorphically in a separable Orlicz sequence space. The reverse question concerning the embedding of an Orlicz sequence space into an Orlicz function space is not yet completely settled. Though we shall not directly discuss this question in Section 4, let us make here some comments concerning it. In an Orlicz function space L_F there are subspaces isomorphic to l_G so that the unit vectors in l_G correspond to functions in L_F which have disjoint supports. These spaces l_G are easy to classify. The situation here is very similar to that of l_G subspaces of an l_H space, and a suitably reformulated version of [9, Th. 1] gives a characterization of all G so that l_G can be embedded into L_F by functions with disjoint supports. There are also subspaces of L_F which are isomorphic to l_G so that the unit vectors in l_G correspond to independent random variables in L_F. These subspaces of L_F were investigated in [1]. The structure of general l_G subspaces of L_F is, however, still unclear, in particular for those functions F whose interval (as defined in Section 4) contains the number 2. It is perhaps worthwhile to point out here the major role played by 2 (or more precisely by the space l_2) in the study of Orlicz function spaces. This is evident from the statements of many results as well as from most proofs. The main reason for this is the fact that in any separable Orlicz function spaces, the Rademacher functions span a subspace isomorphic to l_2. In the theory of Orlicz sequence spaces, on the other hand, the space l_2 plays no special role.

The second result of Section 4 gives a necessary condition for embedding isomorphically one Orlicz function space into another Orlicz function space. Our main interest in this result stems from the following corollary. If L_F is a reflexive Orlicz function space which is isomorphic to L_G and the interval associated with F does not contain 2, then F and G are equivalent Orlicz functions, that is, L_F and L_G consist of the same functions. We do not know whether the restriction concerning 2 is really necessary. This result exhibits a perhaps unexpected difference between Orlicz sequence spaces and Orlicz function spaces. In [8] and [9] (and also Section 3 below) we have exhibited several examples of nonequivalent Orlicz functions which generate isomorphic sequence spaces. Thus, an Orlicz sequence space may have many nonequivalent representations as a symmetric sequence space. On the other hand, a reflexive Orlicz function space L_F (with 2 not contained in the interval associated with F) has a unique representation as a rearrangement-invariant function space on $[0,1]$.

We recall now some definitions concerning Orlicz sequence spaces (basic notions related to Orlicz function spaces will be reviewed in the beginning of Section 4). By an Orlicz function F, we mean a convex continuous strictly increasing function on $[0,\infty)$ such that $F(0)=0$. For the study of Orlicz sequence spaces, only the values of F near 0 are of importance so that quite often we consider the values of F only on $[0,1]$. The function F is said to satisfy the Δ_2 condition (at 0) if $\sup_{0<x\leqq 1} F(2x)/F(x)<\infty$. Unless stated otherwise, we assume that the Orlicz functions appearing in this paper satisfy the Δ_2 condition. For an Orlicz function satisfying the Δ_2 condition (at 0), the Orlicz sequence space l_F consists of all the sequences $x=\{x_i\}_{i=1}^{\infty}$ of reals so that $\sum_{i=1}^{\infty} F(|x_i|)<\infty$. The unit ball of l_F consists of those sequences for which $\sum_{i=1}^{\infty} F(|x_i|)\leqq 1$. Two such functions, F and G are called equivalent (at 0) if $A^{-1}\leqq F(x)/G(x)\leqq A$ for some $A>0$ and all $0<x\leqq 1$. The spaces l_F and l_G consist of the same sequences if and only if F is equivalent to G (at 0). For an Orlicz function F (which satisfies the Δ_2 condition at 0) the set $E_{F,t}=\overline{\{F(sx)/F(s)\}_{0<s\leqq t}}$ is norm compact in $C(0,1)$ for every $t>0$ (the closure is taken in the norm topology of $C(0,1)$). Other norm compact sets in $C(0,1)$ which will be of interest to us are $E_F=\bigcap_{t>0} E_{F,t}$, $C_{F,t}=\overline{\text{conv}}\, E_{F,t}$ and $C_F=\overline{\text{conv}}\, E_F$. All these sets are invariant under the action of the flow T_t; $0<t\leqq 1$, defined by $T_tH(x)=H(tx)/H(t)$. An Orlicz function F is called minimal if $E_{F,1}$ has no proper closed subsets which are invariant under the flow T_t, in other words, if for every $G\in E_{F,1}$ there is a sequence t_i such that $T_{t_i}G$ tends uniformly to F.

A general reference on Orlicz spaces (mainly Orlicz function spaces) is [4]. A detailed exposition of the basic properties of Orlicz sequence spaces is given in [5].

2. Subspaces of Orlicz sequence spaces which are isomorphic to l_p

Let F be an Orlicz function which satisfies the Δ_2 condition at 0. As shown in [8] and [9], the Orlicz functions G such that l_G is isomorphic to a subspace of l_F are exactly those functions which are equivalent to functions in $C_{F,1}$. It was also noted in [8] that the Schauder-Tychonoff fixed point theorem implies that there is always some p such that $x^p \in C_{F,1}$. Our main purpose in this section is to characterize precisely the values of p such that $x^p \in C_{F,1}$, that is those p for which l_p is isomorphic to a subspace of l_F.

We shall show that the set of p's such that $x^p \in C_{F,1}$ coincides with the interval $[\alpha_F, \beta_F]$ associated with F in several places in the literature (see e.g. [10]). The interval is defined by

$$\alpha_F = \sup\{p;\ \sup_{0<x,t\leqq 1} F(tx)/F(t)x^p < \infty\}$$

$$\beta_F = \inf\{p;\ \inf_{0<x,t\leqq 1} F(tx)/F(t)x^p > 0\}.$$

It is clear that for every Orlicz function F satisfying the Δ_2 condition at 0 $1 \leqq \alpha_F \leqq \beta_F < \infty$.

THEOREM 1. *Let F be an Orlicz function satisfying the Δ_2 condition at* 0. *Then the following assertions are equivalent*:

1) $x^p \in C_F$
2) x^p *is equivalent to a function in* $C_{F,1}$
3) l_p *is isomorphic to a subspace of* l_F
4) $p \in [\alpha_F, \beta_F]$.

PROOF. The implication (1) ⇒ (2) is obvious. The equivalence of (2) and (3) was proved in [9]. That (2) ⇒ (4) is also obvious. Indeed, if $p < \alpha_F$ and if $p < r < \alpha_F$ then there is a constant C such that $F(tx) < CF(t)x^r$, $0 < x, t \leqq 1$. Hence for all $G \in C_{F,1}$, $G(x) \leqq Cx^r$, $0 \leqq x \leqq 1$, and thus x^p is not equivalent to any function in $C_{F,1}$. A similar argument applies to the case $p > \beta_F$. The only implication which remains to be proved is (4) ⇒ (1). Our proof of this implication is based on an argument which was suggested by A. Pazy.

If $\alpha_F = \beta_F$ then the above mentioned fixed point theorem proves the desired result. We assume therefore that $\alpha_F < p < \beta_F$ and prove that $x^p \in C_F$. Since C_F is closed, this will show that $x^p \in C_F$ also for $p = \alpha_F$ or $p = \beta_F$. Let $f(x) = F(x)/x^p$, $0 < x \leqq 1$. By our assumption we have $\sup_{0<y\leqq x\leqq 1} f(x)/f(y) = \infty$ and $\inf_{0<y\leqq x\leqq 1} f(x)/f(y) = 0$. Hence, for every n there are $0 < u_n < v_n < w_n < 1$, such that $w_n \to 0$ and

$$nf(u_n) < f(v_n), \quad nf(w_n) < f(v_n). \tag{2.1}$$

Let $a_n = u_n/w_n$, $b_n = v_n/w_n$ and

$$G_n(x) = C_n^{-1} \int_{a_n}^{1} F(tw_nx)t^{-p-1}dt$$

where $C_n = \int_a^1 F(tw_n)t^{-p-1}\, dt$. Clearly $G_n \in C_{F,w_n}$ for every n. By substituting $y = tx$ we get that

$$G_n(x) = C_n^{-1}x^p \int_{a_nx}^{x} F(yw_n)y^{-p-1}dy.$$

Since $\int_{a_nx}^{x} = \int_{a_n}^{1} + \int_{a_nx}^{a_n} - \int_{x}^{1}$, it follows that

$$G_n(x) = x^p + g_n(x) - h_n(x) \tag{2.2}$$

where

$$g_n(x) = C_n^{-1}x^p \int_{a_nx}^{a_n} F(tw_n)t^{-p-1}dt \leqq C_n^{-1}x^{-1}a_n^{-p}F(u_n) \tag{2.3}$$

$$h_n(x) = C_n^{-1}x^p \int_{x}^{1} F(tw_n)t^{-p-1}dt \leqq C_n^{-1}x^{-1}F(w_n). \tag{2.4}$$

Since $b_n/a_n = v_n/u_n \to 0$

$$C_n \geqq \int_{b_n/2}^{b_n} F(tw_n)t^{-p-1}dt \geqq b_n^{-p}F(v_n)/2K \tag{2.5}$$

where K denotes the Δ_2 constant of F.

By (2.1), (2.3) and (2.5),

$$g_n(x) \leqq 2Kb_n^pF(u_n)/(xa_n^pF(v_n)) = 2Kf(u_n)/xf(v_n) \leqq 2K/nx.$$

Similarly by (2.1), (2.4) and (2.5),

$$h_n(x) \leqq 2Kb_n^pF(w_n)/xF(v_n) = 2Kf(w_n)/xf(v_n) \leqq 2K/nx.$$

It follows from (2.2) that $G_n(x) \to x^p$ pointwise and thus, by the compactness of $C_{F,1}$, uniformly on $[0,1]$. Hence $x^p \in C_F$ and this concludes the proof.

Before giving some immediate consequences of the theorem let us make some comments concerning the interval associated with an Orlicz function. Let F be an Orlicz function such that l_F is reflexive. Then, as is well known, $(l_F)^*$ is isomorphic to the Orlicz space l_{F^*} where F^* is defined by

$$F^*(y) = \sup_{0<x} (xy - F(x)).$$

The connection between the interval of F and that of F^* is given by

$$\alpha_F^{-1} + \beta_{F^*}^{-1} = 1, \; \alpha_{F^*}^{-1} + \beta_F^{-1} = 1. \tag{2.6}$$

Indeed, assume that $F(tx)/F(t) \leqq cx^p$ for some constant c and all x and t. Passing to the conjugate functions, we get that $(F(t\,\cdot)/F(t))^*(y) \geqq dy^q$ for some $d > 0$ where $p^{-1} + q^{-1} = 1$. Since

$$(F(t\,\cdot)/F(t))^*(y) = F^*(F(t)yt^{-1})/F(t)$$

and $F^*(F(t)/t)/F(t)$ is bounded away from 0 and ∞, it follows that $F^*(sy)/F^*(s) > ky^q$ for all s and y and some $k > 0$. This proves the first equation in (2.13) and the second follows by duality.

Another remark concerning the interval of F is that it coincides with the one introduced by Lindberg [5]. The definition of Lindberg is as follows. For an Orlicz function F define

$$a_F = \liminf_{x\to 0}\; xF'(x)/F(x), \quad b_F = \limsup_{x\to 0} xF'(x)/F(x).$$

Now put $\hat{a}_F = \sup a_G$, $\hat{b}_F = \inf a_G$ where the sup and inf are taken over all G which are equivalent to F at 0. We claim that for every F, $\alpha_F = \hat{a}_F$ and $\beta_F = \hat{b}_F$. Indeed, a straightforward computation shows that for every F, $a_F \leqq \alpha_F$. Since α_F depends only on the equivalence class of F it follows that $\hat{a}_F \leqq \alpha_F$. To prove the reverse inequality, take a $p < \alpha_F$, and put $g(x) = F(x)/x^p$, $g(0) = 0$. Then g is continuous on $[0,1]$ and it follows from the definition of α_F that $\sup_{x,t} g(xt)/g(t) < \infty$. Let $h(x) = \sup_{0\leqq y<x} g(y)$ and $H(x) = \int_0^x h(t)t^{p-1}dt$. Then H is an Orlicz function equivalent to F and $xH'(x)/H(x) \geqq p$ for all x and thus $a_H \geqq p$. This proves that $\alpha_F = \hat{a}_F$ and that $\beta_F = \hat{b}_F$ is proved similarly. It should be noted that, in general, there is no function G equivalent to F such that $a_G = \alpha_F$ and $b_G = \beta_F$. If, for example, $p = a_G = b_G$ for some G, then E_G consists only of x^p. This is not necessarily the case if we assume only that $p = \alpha_G = \beta_G$ (see Example 1 in the next section).

COROLLARY 1. *Let l_F be a reflexive Orlicz sequence space. Then l_p is isomorphic to a subspace of l_F if and only if l_p is isomorphic to a quotient space of l_F.*

PROOF. This follows from Theorem 1 and (2.6).

COROLLARY 2. *Let F and G be Orlicz functions satisfying the Δ_2 condition at 0. Then every bounded linear operator from l_F into l_G is compact if and only if $\alpha_F > \beta_G$.*

PROOF. The "if" part is well known. The proof of the fact that every operator from l_p into l_r is compact if $p > r$ works just as well here (see [12]; the argument actually goes back to Banach). The "if" part is given in a more general context in Milman [11].

As for the "only if" part, assume that $p = \alpha_F \leqq \beta_G = r$. By Theorem 1 and Corollary 1 there is an operator T_1 from l_F onto l_p and an isomorphism T_2 from l_r into l_G. Let I be the formal identity map from l_p into l_r. Then $T = T_2 I T_1$ is a noncompact operator from l_F into l_G.

3. Minimal Orlicz functions

As is well known, the l_p spaces have a unique symmetric basis up to equivalence. Our first result in this section shows that this property characterizes them among the minimal Orlicz spaces.

THEOREM 2. *Let $F(x)$ be a minimal Orlicz function which is not equivalent to any x^p. Then l_F has uncountably many mutually nonequivalent symmetric bases.*

PROOF. It follows from the definition of minimality and from Pelczynski's decomposition method (cf. [8, p. 389]) that for every $G \in E_{F,1}$, the space l_G is isomorphic to l_F. Hence it will be enough to show that $E_{F,1}$ contains uncountably many mutually nonequivalent functions.

Assume that there are only countably many equivalence classes in $E_{F,1}$ and let G_i be representatives of these classes (the class containing F will be represented by F). For all integers i and k, set

$$A_{i,k} = \{H;\ H \in E_{F,1},\ k^{-1} \leqq H(x)/G_i(x) \leqq k \text{ for } 0 < x \leqq 1\}.$$

The sets $A_{i,k}$ are closed and their union covers $E_{F,1}$. By Baire's category theorem, there is a pair (i, k) such that $A_{i,k}$ contains a (relatively) open set O. By minimality, for every $H \in E_{F,1}$ there is a t such that $H(tx)/H(t) \in O$. Since $H(x)$ is equivalent to $H(tx)/H(t)$ it follows that $E_{F,1}$ consists of only one equivalence class, i.e., all the functions in $E_{F,1}$ are equivalent to F.

For every $0 < t \leqq 1$ set $B_t = \{H; H \in E_{F,1}, H(tx)/H(t) \in O\}$. Then B_t is open and again by minimality, $\bigcup_{0<t\leqq 1} B_t$ covers $E_{F,1}$. By the compactness of $E_{F,1}$ there is a $u > 0$ such that $\bigcup_{u \leqq t \leqq 1} B_t = E_{F,1}$. It follows that for every $0 < s \leqq 1$ there is a $u \leqq t < 1$ such that $F(stx)/F(st) \in O$, i.e., $k^{-2} \leqq F(stx)/F(st)F(x) \leqq k^2$, $0 < x \leqq 1$. Since $t \geqq u$, the Δ_2 condition implies that there is some constant $c > 0$ such that for $0 < s, x \leqq 1$, $c^{-1} \leqq F(sx)/F(s)F(x) \leqq c$. By [13, problem 99] it follows that $F(x)$ is equivalent to x^p for some p, contrary to our assumption.

REMARK. The concept of a minimal Orlicz space can be generalized in a natural way to the setting of general symmetric bases in view of [9, Th. 4]. A symmetric basis $\{e_i\}_{i=1}^{\infty}$ of a Banach space X is said to be *minimal symmetric* if every sequence $\{u_j\}$ of the form $u_j = \sum_{i=p_j+1}^{p_{j+1}} e_i$ with $p_1 < p_2 < \cdots$, spans a subspace of X which is isomorphic to X. It is therefore natural to ask the following question. *Assume that the Banach space X has up to equivalence a unique symmetric basis and that this basis is minimal symmetric. Is X isomorphic to c_0 or to some l_p?*

We now describe a general method of representing Orlicz functions M in a form in which the set $E_{M,1}$ can be easily described. Our main application of this representation is in producing some examples of minimal functions. Let $F(x)$ and $G(x)$ be two strictly increasing continuous convex functions on $[e^{-1},1]$† such that

$$F(1) = G(1) = 1,$$

$$xF'(x)/F(x) \geqq F'(1) = G'(1) \leqq xG'(x)/G(x), \quad x \in [e^{-1} \;\; 1],$$

and

$$F(e^{-1}) = \exp(-p_1), \; G(e^{-1}) = \exp(-p_2), \text{ with } p_1 < p_2.$$

For every sequence of digits $\theta = \{\theta(i)\}_{i=1}^{\infty}$ with $\theta(i)$ equal to 0 or 1, for each i we define an Orlicz function M_θ on $[0,1]$ by putting $M_\theta(1) = 1$, $M_\theta(0) = 0$ and for $\exp(-i) \leqq t < \exp(-i+1)$, $i = 1,2,\cdots$

$$M_\theta(t) = \begin{cases} M_\theta(\exp(-i+1))F(t\exp(i-1)) & \text{if } \theta(i) = 0 \\ M_\theta(\exp(-i+1))G(t\exp(i-1)) & \text{if } \theta(i) = 1. \end{cases}$$

It is easy to check (cf. [8, Lemma 2]) that for every θ, M_θ is an Orlicz function satisfying the Δ_2 condition.

Let us list some simple properties of the functions M_θ. The proof of these observations is straightforward.

i) $M_\theta(\exp(-k)) = \exp(-kp_1 - (p_2 - p_1)\sum_{i=1}^{k}\theta(i))$ for $k = 1,2,\cdots$. It follows in particular that up to equivalence, M_θ is determined by p_1, p_2 and θ and does not depend on the special choice of F and G.

ii) For two sequences $\theta = \{\theta(i)\}$ and $\eta = \{\eta(i)\}$, the function M_θ is equivalent to M_η if and only if $\sup_k \left| \sum_{i=1}^{k}\eta(i) - \sum_{i=1}^{k}\theta(i) \right| < \infty$.

† We chose e^{-1} simply because of the typographical convenience in writing $\exp(k) = e^k$.

iii) For fixed F and G, the set of all the functions of the form M_θ is a norm compact set in $C(0,1)$. The map $\theta \to M_\theta$ is a homeomorphism from $\{0,1\}^{\aleph_0}$ with the product topology into $C(0,1)$.

iv) Let T be the map defined by $TH(x) = H(e^{-1}x)/H(e^{-1})$. Then $TM_\theta = M_{S\theta}$ where $S\theta(i) = \theta(i+1)$ (i.e. S is the shift by one to the left).

v) $E_{M_\theta,1}$ consists of functions equivalent to functions of the form M_η where η is a limit (in the topology of pointwise convergence) of sequences of the form $\{S^{n_j}\theta\}$, i.e. η is such that for every k there exists an $n = n(k)$ such that $\eta(i) = \theta(n+i)$, $i = 1,\cdots,k$. Conversely, every such M_η belongs to $E_{M_\theta,1}$.

Some further properties of M_θ which are of interest in the study of E_{M_θ} are given in Propositions 1 and 2.

PROPOSITION 1. *Let p_1, p_2, θ and M_θ be as above. Then*

$$\alpha_{M_\theta} = p_1 + (p_2 - p_1)\liminf_{k\to\infty} k^{-1}(\inf_n \sum_{i=n+1}^{n+k} \theta(i)) \tag{3.1}$$

$$\beta_{M_\theta} = p_1 + (p_2 - p_1)\limsup_{k\to\infty} k^{-1}(\sup_n \sum_{i=n+1}^{n+k} \theta(i)). \tag{3.2}$$

PROOF. It is clear, by the Δ_2 condition, that in the definition of α_F for a general Orlicz function F it is enough to consider only expressions of the form $F(\exp(-n-k))/F(\exp(-n))\exp(-kp)$. In the case of $F = M_\theta$ this expression becomes (by observation (i))

$$\exp\Big(-kp_1 - (p_2 - p_1)\sum_{i=n+1}^{n+k} \theta(i) + kp\Big).$$

Hence α_{M_θ} is the sup of all the p's for which

$$\sup_{n,k} k(p - p_1 - (p_2 - p_1)k^{-1} \sum_{i=n+1}^{n+k} \theta(i)) < \infty.$$

This implies (3.1). The proof of (3.2) is similar.

PROPOSITION 2. *Let θ and M_θ be as above. M_θ is equivalent to a minimal Orlicz function if and only if there is a constant K such that for every integer k there is an integer $n = n(k)$ with the following property: For every integer s there is an $m \leqq n$ such that*

$$\left| \sum_{i=s+m+1}^{s+m+j} \theta(i) - \sum_{i=1}^{j} \theta(i) \right| \leqq K \quad for \quad j = 1,2,\cdots,k.$$

PROOF. Assume first that such a K exists. Let $n_j \to \infty$ be a sequence of integers for which $\eta = \lim S^{n_j}\theta$ exists. Since any block of digits appearing in η appears also in θ, our assumption will show that for every k there is an $l = l(k)$ such that

$$\left| \sum_{i=l+1}^{l+j} \eta(i) - \sum_{i=1}^{j} \theta(i) \right| \leqq K \quad \text{for} \quad j = 1,2,\cdots,k.$$

It follows from observation (ii) that for any limiting point τ of the sequence $\{S^{l(k)}\eta\}_{k=1}^{\infty}$ the function M_τ is equivalent to M_θ. By observations (iv) and (v) it follows that for every $N \in E_{M_\theta,1}$ there is an $\tilde{M} \in E_{N,1}$ with $\tilde{M}$ equivalent to M_θ. Taking in particular N to be minimal, we get a minimal function $\tilde{M}$ equivalent to M_θ.

Now we prove the converse. Assume that there is a minimal function N such that $A^{-1} \leqq N(x)/M_\theta(x) \leqq A$ for some A. Assume also that for every integer m there is an integer $k = k(m)$ and sequences $\{s_1(m,n)\}_{n=1}^{\infty}$ and $\{s_2(m,n)\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} [s_2(m,n) - s_1(m,n)] = \infty,$$

and for every $s_1(m,n) \leqq s < s_2(m,n)$ there is a $j \leqq k(m)$ with

$$\left| \sum_{i=j+1}^{s+j} \theta(i) - \sum_{i=1}^{j} \theta(i) \right| \geqq m.$$

Fix m and let η_m be any limit point of the sequence $\{S^{s_1(m,n)}\theta\}_{n=1}^{\infty}$. Then $M_{\eta_m} \in E_{M_\theta,1}$ and for every integer t there is a $j \leqq k(m)$ such that

$$\left| \sum_{i=t+1}^{t+j} \eta_m(i) - \sum_{i=1}^{j} \theta(i) \right| \geqq m.$$

It follows from observations (i), (ii) and (v) that for any $M \in E_{M_{\eta_m},1}$ there is an $0 < x \leqq 1$ such that $M(x)/M_\theta(x)$ is outside the interval $[B\exp(-m(p_2 - p_1)), B^{-1}\exp(m(p_2 - p_1))]$, where B is a constant depending on the Δ_2 constant of M but not on m. This however contradicts the minimality of N if $A^4 < B^{-1}\exp(m(p_2 - p_1))$.

Now we consider two examples. Example 1 was already described in [8] and [9, Example 3] (the only difference is that here we have replaced 2 by e).

EXAMPLE 1. We construct a function M_θ where θ is a sequence defined by induction simultaneously with another sequence η as follows: $\theta(1) = 1, \eta(1) = 0$ and for $n = 0,1,2,\cdots$

$\theta(2^{3n}+i)=\theta(i),\ 1 \leqq i \leqq 2^{3n};$ $\theta(2^{3n+1}+i)=\theta(i),\ 1 \leqq i \leqq 2^{3n+1};$

$\theta(2^{3n+2}+i)=\eta(i),\ 1 \leqq i \leqq 2^{3n+2};$ $\eta(2^{3n}+i)=\theta(i),\ 1 \leqq i \leqq 2^{3n};$

$\eta(2^{3n+1}+i)=\eta(i),\ 1 \leqq i \leqq 2^{3n+1};$ $\eta(2^{3n+2}+i)=\eta(i),\ 1 \leqq i \leqq 2^{3n+2}.$

Thus the sequences begin as follows:

$$\theta = \{1,1,1,1,0,1,0,1,1,1,1,1,0,1,0,1,\cdots\}$$

$$\eta = \{0,1,0,1,0,1,0,1,1,1,1,1,0,1,0,1,\cdots\}.$$

We shall prove that M_θ is equivalent to a minimal function, that it is not equivalent to any x^p and that its interval consists of the single point $p_1 + 2(p_2-p_1)/3$, despite the fact that there exists no Orlicz function M equivalent to M_θ for which $\lim_{x\to 0} xM'(x)/M(x)=p_1+2(p_2-p_1)/3$. (The first two claims have already been proved in [9]. Here we repeat the proofs in the present terminology.)

For every n, let A_n(resp. B_n) be the block of the first 2^{3n} digits in θ(resp. η). By the definition of η, both A_{n+1} and B_{n+1} contain a block equal to A_n. Since θ can be written as $\theta = C_1C_2C_3\cdots$ where each block C_j is equal either to A_{n+1} or to B_{n+1}, it follows that every block of θ of length $\geqq 3\cdot 2^{3n+3}$ contains in it either A_{n+1} or B_{n+1} and thus the block A_n. Thus the condition in the statement of Proposition 2 is satisfied with $K=0$ and $n(k)=3\cdot 2^{3n+3}$ for $k \leqq 2^{3n}$. This proves that M_θ is equivalent to a minimal function.

Clearly $\eta = \lim_n S^{2^{3n+2}}\theta$ and $\theta = \lim_n S^{2^{3n}}\eta$; hence, $M_\eta \in E_{M_\theta,1}$ and $M_\theta \in E_{M_\eta,1}$, On the other hand, since $\sum_{i=1}^{2^{3n}}(\theta(i)-\eta(i)) = 2^n$ it follows that M_θ is not equivalent to M_η while l_{M_θ} is isomorphic to l_{M_η}. Thus M_θ is not equivalent to any x^p.

In order to compute the interval of M_θ let us denote by a_n (resp. b_n) the number of 1's in the block A_n (resp. B_n). Then $a_0 = 1$, $b_0 = 0$ and

$$a_{n+1} = 6a_n + 2b_n,\ b_{n+1} = 4a_n + 4b_n \qquad n = 0,1,2,\cdots.$$

Easy computations show that $\lim_n a_n 2^{-3n} = \lim_n b_n 2^{-3n} = 2/3$. Let $\varepsilon > 0$ be given and choose n such that $|a_n 2^{-3n} - 2/3| < \varepsilon$ and $|b_n 2^{-3n} - 2/3| < \varepsilon$. Notice that $a_n 2^{-3n}(b_n 2^{-3n})$ is the density of 1's in A_n, resp. B_n; thus in any block from $k2^{3n}$ to $(k+1)2^{3n}$, the density of the 1's differs from 2/3 by at most ε. Hence if we take any block in θ of length $l\cdot 2^{3n}$ then the density of the 1's in it is between $(l-2)(2/3-\varepsilon)/l$ and $((l-2)(2/3+\varepsilon)+2)/l$. Therefore, for sufficiently large l the density is between $2/3-2\varepsilon$ and $2/3+2\varepsilon$. It follows from (3.1) and (3.2) that $\alpha_{M_\theta} = \beta_{M_\theta} = p_1 + 2(p_2-p_1)/3$. We have thus an example of a nontrivial minimal Orlicz function whose interval is a single point.

Our second example will be of a minimal function whose interval is non-degenerate.

EXAMPLE 2. Let $\{n_j\}$ be a sequence of positive integers such that $\sum_{j=1}^{\infty} n_j^{-1} \leqq 1/3$. We define two sequences θ and η of 0's and 1's as follows: Let $m_j = n_1 \cdot n_2, \cdots, n_{j-1}$ $(m_1 = 1)$ and let A_j (resp. B_j) denote the block of the first m_j digits of θ (resp. η). A_1 consists of the digit 1 and B_1 consists of the digit 0. For $j > 1$, A_j and B_j are defined inductively by

$$A_{j+1} = A_j A_j \cdots A_j B_j \quad (A_j \text{ appearing } n_j - 1 \text{ times})$$

$$B_{j+1} = B_j B_j \cdots B_j A_j \quad (B_j \text{ appearing } n_j - 1 \text{ times}).$$

The same argument as in Example 1 shows that for this θ, M_θ is a minimal Orlicz function. The condition in the statement of Proposition 2 is satisfied with $K = 0$ and $n(m_{j-1}) = 3m_j$. By the choice of the n_j the density of the 1's in A_j is larger than $\prod_{i=1}^{j-1}(1 - n_i^{-1}) \geqq 2/3$ while the density of the 1's in B_j is less than 1/3. It follows from Proposition 1 that $\alpha_{M_\theta} \leqq p_1 + (p_2 - p_1)/3$, $\beta_{M_\theta} \geqq p_1 + 2(p_2 - p_1)/3$.

In spite of the fact that the space l_{M_θ} has subspaces isomorphic to l_p for an entire interval of p's, it does not have any l_p as a complemented subspace (a similar result for Example 1 was proved in [9]). In order to prove this, we apply [9, Th. 2] and show that x^p is strongly nonequivalent to $E_{M_\theta,1}$ for every p. For the sake of simplicity we shall prove this assertion only under the assumption that the n_j do not grow too fast, say if $n_j \leqq m_j = n_1 n_2 \cdots n_{j-1}$.

By the definition of θ any block of digits in θ of length $(n_j + 2)m_j$ has sublocks equal to A_j and to B_j. This means that for every integer k there are integers s and u ; $s, u \leqq (n_j + 1)m_j$ such that

$$M_\theta(x\exp(k-s))/M_\theta(\exp(-k-s)) = M_\theta(x), \ \exp(-m_j) \leqq x \leqq 1 \tag{3.3}$$

$$M_\theta(x\exp(-k-u))/M_\theta(\exp(-k-u)) = M_\eta(x), \ \exp(-m_j) \leqq x \leqq 1. \tag{3.4}$$

Consider now the $(n_j + 2)m_j$ points $x_i = \exp(-i)$, $i = 1, \cdots, (n_j + 2)m_j$. Assume that there is an integer k and a constant K such that for $i = 1, 2, \cdots, (n_j + 2)m_j$ and $x_i = \exp(-i)$

$$\begin{aligned} K^{-1} &\leqq M_\theta(x_i \exp(-k))/x_i^p M_\theta(\exp(-k)) \\ &= M_\theta(\exp(-i-k))/\exp(-ip)M_\theta(\exp(-k)) \leqq K. \end{aligned} \tag{3.5}$$

Let s be such that (3.3) holds for this k. By dividing (3.5) with $i = s + m_j$ by (3.5) with $i = s$ and then applying (3.3) we get

(3.6) $$K^{-2} \leqq M_\theta(\exp(-m_j))/\exp(-m_jp) \leqq K^2.$$

Similarly it follows from (3.4) and (3.5) that

(3.7) $$K^{-2} \leqq M_\eta(\exp(-m_j))/\exp(-m_jp) \leqq K^2.$$

It follows from (3.6) and (3.7) that

(3.8) $$K^{-4} \leqq M_\theta(\exp(-m_j))/M_\eta(\exp(-m_j)) \leqq K^4.$$

However, since the density of 1's in A_j is $> 2/3$ and in B_j is $> 1/3$, it follows (see observation (i)) that

(3.9) $$M_\theta(\exp(-m_j))/M_\eta(\exp(-m_j)) \leqq \exp(-(p_2 - p_1)m_j/3).$$

By (3.8) and (3.9) $\log K > (p_2 - p_1)m_j/12$. Since the number of points x_i we used is $\leqq 2n_jm_j \leqq 2m_j^2$ we proved, as desired, the fact that x^p is strongly non-equivalent to $E_{M_\theta,1}$.

Let us now describe all possible intervals of minimal Orlicz functions. Actually, there is only one limitation: if $\alpha = 1$ and F is minimal then $F(x) = x$. Indeed if $t_nF'(t_n)/F(t_n) \to 1$ then the convexity of F implies that $F(t_nx)/F(t_n) \to x$, for $0 \leqq x \leqq 1$ and hence by minimality $F(x) = x$. However for every $1 < \alpha \leqq \beta < \infty$ there is a minimal Orlicz function whose interval is exactly $[\alpha, \beta]$. This is a consequence of the constructions given in Examples 1 and 2 and the following remark. Let $p > 1$ and let $t > (p-1)/p$. The functions $F(x) = x^p$ and $G(x) = px - p + 1$ satisfy in the interval $[t, 1]$ the assumptions which appear in the definition of M_θ (with t replacing e^{-1}). For every $r > p$ we can of course choose t such that $G(t) = t^r$.

To conclude this section, let us justify the remark previously made that the functions M_θ represent essentially all Orlicz functions: *Every Orlicz function M such that l_M is reflexive is equivalent to a function of the form M_θ (if in the definition of M_θ we replace e^{-1} by a suitable $t \in (0,1)$).* Indeed, since l_M is reflexive, we can assume with no loss of generality that for some $1 < p < r \leqq \infty$ and all $x \in (0,1)$, $p \leqq xM'(x)/M(x) \leqq r$, and hence $t^r \leqq M(tx)/M(x) \leqq t^p$, $0 < x, t \leqq 1$. Choose now t, F and G as in the preceding paragraph so that $F(t) = t^p$, $G(t) = t^r$.

Now we construct inductively a sequence $\theta = \{\theta(i)\}$ as follows: $\theta(1) = 1$ and if $M_\theta(t^n)t^p \leqq M(t^{n+1})$ then we set $\theta(n+1) = 0$; otherwise, $\theta(n+1) = 1$. It can be easily verified that

$$M_\theta(t^n) \leqq M(t^n) \leqq t^{p-r}M_\theta(t^n); \quad n = 1, 2, \ldots.$$

Clearly, M_θ is equivalent to M.

4. Orlicz function spaces

First we give some basic definitions. Let F be an Orlicz function on $[0,\infty]$. By L_F we denote the space of all measurable functions f on $[0,1]$ such that $\int_0^1 F(\lambda|f(t)|)dt < \infty$ for some $\lambda > 0$. We say that F satisfies the Δ_2 condition at ∞ if $\sup_{x\geqq 1}F(2x)/F(x)<\infty$. This supremum is called the Δ_2 constant of F (at ∞). If F satisfies the Δ_2 condition at ∞, L_F consists of all the measurable f such that $\int_0^1 F(|f(t)|)dt < \infty$. The unit ball of L_F is taken as $\{f;\ \int_0^1 F(|f(t)|)dt \leqq 1\}$. Unless stated otherwise, we shall assume whenever we consider L_F that F satisfies the Δ_2 condition at ∞. Up to isomorphism, L_F is determined by the values of $F(x)$ for large x.

If we replace the interval $[0,1]$ on which we integrate $F(|f(t)|)$ by an arbitrary subset of the line with a finite positive measure and consider the function space on this set, we clearly do not get anything new. On the other hand, if we replace $[0,1]$ by the whole line (or any set of infinite measure), new features enter into the study of Orlicz function spaces. In particular, the values of F near ∞ as well as near 0 are important. The case of general Orlicz function spaces (which certainly deserves careful study) is not treated here.

The interval associated with an Orlicz function F at ∞ is denoted by $[\alpha_F^\infty, \beta_F^\infty]$ and is defined by

$$\alpha_F^\infty = \sup\{p;\ \sup_{1\leqq x,y} F(x)y^p/F(xy) < \infty\}$$

$$\beta_F^\infty = \inf\{p;\ \inf_{1\leqq x,y} F(x)y^p/F(xy) > 0\}.$$

In studying the connection between Orlicz function spaces and Orlicz sequence spaces, it is convenient to associate with F some subsets of $C(0,1)$. The set of functions $G(x)$ which are of the form $\lim_{n\to\infty} F(xy_n)/F(y_n)$, $0 \leqq x < 1$, for some sequence $y_n \to \infty$ is denoted by E_F^∞. Notice that even if F is defined only for large x, the limit is defined for every $x > 0$. We always have $G(0) = 0$. The closed convex hull of E_F^∞ is denoted by C_F^∞. If F satisfies the Δ_2 condition at ∞ then E_F^∞ and C_F^∞ are nonempty compact subsets of $C(0,1)$. An argument similar to that given in the proof of Theorem 1 shows that $x^p \in C_F$ if and only if $p \in [\alpha_F^\infty, \beta_F^\infty]$.

We pass now to our first result on function spaces.

THEOREM 3. *An Orlicz function space L_F which is not isomorphic to a Hilbert space, is not isomorphic to a subspace of a separable Orlicz sequence space l_G.*

PROOF. If F does not satisfy the Δ_2 condition at ∞, L_F is nonseparable and there is nothing to prove. We therefore assume that F satisfies Δ_2 at ∞.

Let $r_n(t) = \operatorname{sign}\sin 2^{n-2}\pi t$, $n = 1,2,\cdots$ be the Rademacher functions on $[0, 1]$. The well-known Khintchin inequality implies that the span of the $\{r_n\}_{n=1}^{\infty}$ in L_F is isomorphic to l_2, i.e. that for some constant K

$$K^{-1}(\sum_n \lambda_n^2)^{\frac{1}{2}} \leqq \Big\| \sum_n \lambda_n r_n \Big\|_F \leqq K(\sum_n \lambda_n^2)^{\frac{1}{2}} \tag{4.1}$$

for every choice of $\{\lambda_n\}$. The constant K in (4.1) can be chosen to depend only on the Δ_2 constant of F, if F is normalized by $F(1) = 1$ (since this Δ_2 constant determines constants C and p so that $F(x) \leqq Cx^p$, $x \geqq 1$). It follows that we may assume that (4.1) holds with the same K if F is replaced by $F(xy)/F(y)$ for some y. Applying this remark to the case where y_m is chosen to satisfy $F(y_m) = m$ with m being an integer, we get for all choices of $\{\lambda_n\}$ that

$$K^{-1}(\sum_n \lambda_n^2)^{\frac{1}{2}} \leqq \Big\| \sum \lambda_n r_{1,n}^m \Big\| \leqq K(\sum_n \lambda_n^2)^{\frac{1}{2}}; \quad m = 1,2,\cdots \tag{4.2}$$

where $r_{1,n}^m$ are the normalized Rademacher functions on $[0, m^{-1}]$, i.e. $r_{1,n}^m(t) = r_n(mt)/y_m$ if $t < m^{-1}$ and $= 0$ if $t > m^{-1}$. By translating $\{r_{1,n}^m\}_{n=1}^{\infty}$ by $(i-1)/m$, we get the normalized Rademacher sequence $\{r_{i,n}^m\}_{n=1}^{\infty}$ on the interval $[(i-1)/m, i/m]$. Trivially (4.2) remains valid if we replace the index 1 in $r_{1,n}^m$ by some fixed i ($1 \leqq i \leqq m$). Observe also that $\| r_{i,n}^m \| = 1$ for all i, m and n.

Before proceeding with the proof let us recall two simple facts concerning Orlicz sequence spaces which will be needed below.

a) Let H_1 and H_2 be two Orlicz functions on $[0,1]$ with $H_1(1) = H_2(1) = 1$. Assume that the identity map I from l_{H_1} onto l_{H_2} is an isomorphism. Then $C^{-1} \leqq H_1(x)/H_2(x) \leqq C$, $0 < x \leqq 1$ for some constant C depending only on $\| I \| \, \| I^{-1} \|$ and the Δ_2 constants of H_1 and H_2.

b) Let $H(x)$ be an Orlicz function and let $x = \sum_j \gamma_j e_j \in l_H$ (as usual, the e_j denote the unit vectors). Assume that $C_0^{-1} \leqq \sum_j H(|\gamma_j|) \leqq C_0$ for some constant C_0. Then there is a constant D, depending on C_0 and the Δ_2 constant of H such that $D^{-1} \leqq \| x \| \leqq D$.

Assume now that there is an isomorphism T from L_F into l_G for some G satisfying the Δ_2 condition at 0, and let m be an integer. Since $w - \lim_n r_{i,n}^m = 0$ for $1 \leqq i \leqq m$, we may, by a standard procedure, choose subsequences $\{n_{i,k}\}_{k=1}^{\infty}$, $1 \leqq i \leqq m$, of integers and vectors $x_{i,k}$ in l_G such that all $x_{i,k}$ have disjoint supports and

$$\| x_{i,k} - Tr^m_{i,n_{i,k}} \| \leqq 2^{-k}, \quad 1 \leqq i \leqq m; \quad k = 1,2,\cdots. \tag{4.3}$$

The assertion that $x_{i,k}$ have disjoint supports means that

$$x_{i,k} = \sum_{j \in A_{i,k}} \gamma_j e_j, \quad A_{i_1,k_1} \cap A_{i_2,k_2} \neq \emptyset \Rightarrow i_1 = i_2 \text{ and } k_1 = k_2. \tag{4.4}$$

(The $x_{i,k}$, $A_{i,k}$ and γ_j depend also on m. For simplicity of notation we did not write this explicitly.) Consider now the Orlicz functions $H_{i,k}(x) = \sum_{j \in A_{i,k}} G(|\gamma_j| x)$. It follows from (4.3) that $\{H_{i,k}(1)\}$ is bounded and bounded away from 0 by constants independent of m. Since G satisfies Δ_2, the set $\{H_{i,k}(x)\}$ is totally bounded in $C(0,1)$ and hence, without loss of generality we may assume that $H_i(x) = \lim_{k \to \infty} H_{i,k}(x)$ exists uniformly on $[0,1]$ for $1 \leqq i \leqq m$ (otherwise pass to a suitable subsequence). From (4.2), (4.3) and (4.4) it follows that the identity map I_i from l_{H_i} into l_2 is an isomorphism with $\| I_i \| \, \| I_i^{-1} \|$ bounded by a constant independent on i and m. Hence by remark (a) above there is a constant C independent of i and m such that

$$C^{-1} \leqq H_i(x)/x^2 \leqq C, \qquad 0 \leqq x \leqq 1. \tag{4.5}$$

Let D be constant, given in remark (b) above, which corresponds to the function G and the constant $C_0 = 2C$.

For every $1 \leqq i \leqq m$ choose an integer $k(i)$ such that

$$2^{-k(i)} \leqq 1/2mD, \quad |H_{i,k(i)}(x) - H_i(x)| \leqq 1/2mC; \quad 0 \leqq x \leqq 1. \tag{4.6}$$

For simplicity of notation, put $\phi_i = r^m_{i\, r_{i},n_{i},k(i)}$, $y_i = x_{i,k(i)}$ and $B_i = A_{i,k(i)}$. Let $\{\lambda_i\}_{i=1}^m$ be numbers such that $\sum_{i=1}^m \lambda_i = 1$. By (4.3) and (4.6)

$$\| T(\sum_{i=1}^m \lambda_i \phi_i) - \sum_{i=1}^m \lambda_i y_i \| \leqq \sum_{i=1}^m |\lambda_i|/2mD \leqq 1/2D. \tag{4.7}$$

Also, by (4.6)

$$|\sum_{i=1}^m \sum_{j \in B_i} G(|\lambda_i \gamma_j|) - \sum_{i=1}^m H_i(|\lambda_i|)| \leqq m/2mC = 1/2C. \tag{4.8}$$

By (4.5) and (4.8)

$$1/2C \leqq C^{-1} \sum_{i=1}^m \lambda_i^2 - 1/2C \leqq \sum_{i=1}^m \sum_{j \in B_i} G(|\lambda_i \gamma_j|) \leqq C \sum_{i=1}^m \lambda_i^2 + 1/2C \leqq 2C.$$

It follows from the choice of D that

$$D^{-1} \leqq \| \sum_{i=1}^m \lambda_i y_i \| \leqq D,$$

and hence by (4.7)

$$(4.9) \qquad (1/2D) \| T \| \leqq \| \sum_{i=1}^{m} \lambda_i \Phi_i \| \leqq 2D \| T^{-1} \|.$$

Each $\varnothing_i$ is a function with constant absolute value on $[(i-1)/m, i/m]$ and is 0 outside this interval. Since $\|f\| = \| |f| \|$ for every $f \in L_F$, it follows from (4.9) that for every m there is an operator U_m from the m-dimensional Hilbert space into the subspace of L_F consisting of the functions which are constant on each interval of the form $[(i-1)m, i/m)$, such that $\|U_m\| \|U_m^{-1}\|$ is bounded by a constant independent of m. In the terminology of [7] this means that L_F is a $\mathscr{L}_2$ space, i.e. it is isomorphic to a Hilbert space. Q. E. D.

REMARK. The same proof works if we replace L_F by a rearrangement-invariant function space on $[0,1]$ provided that the normalized Rademacher functions $\{r_{i,n}^m\}_{n=1}^{\infty}$ span a subspace isomorphic to l_2 with an isomorphism constant independent of m (it is clearly independent of i). Sufficient conditions for this to happen can be deduced from the interpolation theorem of Semenov [14]. On the other hand, in Theorem 3 we cannot replace the Orlicz space l_G by an arbitrary space with a symmetric basis. In fact, those Orlicz function spaces which have an unconditional basis (i.e. the reflexive Orlicz function spaces, cf. [2]) can be embedded isomorphically into Banach spaces with symmetric bases (cf. [6]).

Let F be an Orlicz function and let $\varepsilon > 0$. The following sets are of importance in the study of L_F:

$$A_\varepsilon^F = \{f; f \in L_F,\ \mu\{t; |f(t)| \geqq \varepsilon \|f\|\} \geqq \varepsilon\}$$

where μ denotes the Lebesgue measure. We prove next a simple generalization of a result of Kadec and Pelczynski [3].

PROPOSITION 3. *Let $\{f_n\}$ be a sequence of functions with norm 1 in L_F. If the set $\{f_n\}$ is not contained in A_ε^F for some $\varepsilon > 0$, then there is a subsequence of $\{f_n\}$ which is equivalent to the unit vector basis in l_G for some $G \in C_F^\infty$.*

PROOF. A rather well-known procedure (see [3] for details) shows that if for every $\varepsilon > 0$ there is an $n = n(\varepsilon)$ with $f_n \notin A_\varepsilon^F$, then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ consisting of functions having "almost disjoint" supports. That is, there exist $g_k \in L_F$ such that $\| g_k - f_{n_k} \| \leqq 2^{-k}$ for every k, and the sets $A_k = \{t; g_k(t) \neq 0\}$ are mutually disjoint. Since $\mu(A_k) \to 0$ there is no loss of generality in assuming that $y_k = \inf\{|g_k(t)|, t \in A_k\}$ tends to ∞ (if we replace $g_k(t)$ by 0 whenever $|g_k(t)|$ is smaller than a suitably chosen y_k, we change g_k only by a small amount in the norm of L_F). Since $\|f_n\| = 1$ we may assume also that $\|g_k\| = 1$ for all k. Consider the functions $H_k(x) = \int_0^1 F(x|g_k(t)|)dt$, $0 \leqq x \leqq 1$. Since $H_k(1) = 1$ for every k, it follows that $H_k(x)$ is in the convex hull of $\{F(xy)/F(y)\}_{y \geqq y_k}$. Since the $\{H_k(x)\}$ form a totally bounded set in $C(0,1)$, there is a subsequence $\{H_{k_j}\}$ of $\{H_k\}$ which converges to some $G \in C_F^\infty$. We assume, as we may, that $|H_{k_j}(x) - G(x)| \leqq 2^{-j}$, $0 \leqq x \leqq 1$. It easily follows from this that the unit vector basis of l_G is equivalent to the sequence $\{g_{k_j}\}$ and thus to the subsequence $\{f_{n_{k_j}}\}$ of the given sequence in L_F.

PROPOSITION 4. *Let F and G be Orlicz functions. Then, there is an isomorphism T from l_G into L_F which takes the unit vector basis in l_G into functions in L_F with mutually disjoint supports if and only if G is equivalent to a function in C_F^∞. If $G \in E_F^\infty$ then there exists such an isomorphism T for which, in addition, Tl_G is complemented in L_F.*

PROOF. The "only if" part has already been proved in the proof of Proposition 3. The proof of the "if" part of the first sentence is identical to the proof of [9, Th. 1]. If $G \in E_F^\infty$ then the isomorphism T which is obtained in that proof maps the unit vectors of l_F into normalized characteristic functions of disjoint subsets of [0,1]. Hence there is a conditional expectation which projects L_F onto Tl_G.

COROLLARY. *Let F be an Orlicz function with $\alpha_F^\infty \geqq 2$. Then l_p is isomorphic to a subspace of L_F if and only if either $p = 2$ or $p \in [\alpha_F^\infty, \beta_F^\infty]$.*

PROOF. Assume that L_F has a subspace X isomorphic to l_p. We observe first that if $X \subset A_\varepsilon^F$ for some $\varepsilon > 0$, then $p=2$. Indeed since $\alpha_F^\infty \geqq 2$ there is for any $r < 2$ a constant K_r such that $F(x) \geqq K_r x^r$ for $x \geqq 1$. Hence for every $\varepsilon > 0$ and every $r < 2$ there is a constant $C = C(r, \varepsilon)$ such that for $f \in A_\varepsilon^F$, $C^{-1}\|f\|_r \leqq \|f\|_F \leqq C\|f\|_r$. Thus our given X is isomorphic to a subspace of L_r for every $r < 2$ and this shows that $p = 2$. If there is no $\varepsilon > 0$ for which $X \subset A_\varepsilon^F$ then by Proposition 3, X has a subspace isomorphic to l_G for some $G \in C_F^\infty$. In this case we get that $p \in [\alpha_F^\infty, \beta_F^\infty]$.

Conversely, if $p \in [\alpha_F^\infty, \beta_F^\infty]$ then $x^p \in C_F^\infty$ and hence, by Proposition 4, l_p is isomorphic to a subspace of L_F. A subspace of L_F isomorphic to l_2 is obtained by taking the Rademacher functions.

We consider now the question of embedding one Orlicz function space into another. Several interesting sufficient conditions were given by Bretagnolle and Dacunha-Castelle [1]. We give here a necessary condition.

THEOREM 4. *Let F and G be Orlicz functions. Assume that* $\beta_F^\infty < 2$ *and that* L_G *is isomorphic to a subspace of* L_F. *Then* $\sup_{x \leq 1} F(x)/G(x) < \infty$.

PROOF. We shall denote the norms in L_F (resp. L_G) by $\|\cdot\|_F$ (resp. $\|\cdot\|_G$). Let m be an integer and denote by $\phi_{i,m}$ the characteristic function of $[(i-1)/m, i/m]$, $i = 1,2,\cdots,m$. Put $\gamma_m = \|\phi_{i,m}\|_G$, i.e. $G(\gamma_m^{-1}) = m$. Let T be an isomorphism from L_G into L_F and put $f_{i,m} = \gamma_m^{-1} T\phi_{i,m}$. Then $\|T^{-1}\|^{-1} \leqq \|f_{i,m}\|_F \leqq \|T\|$ and hence, since F satisfies Δ_2 at ∞, we get that

$$C^{-1} \leqq \int_0^1 F(|f_{i,m}(t)|)dt \leqq C, \qquad 1 \leqq i \leqq m \qquad m = 1,2,\cdots \tag{4.10}$$

for some constant C independent of i and m. For every choice of signs θ_i ($\theta_i = \pm 1$) we have

$$\gamma_m^{-1}\|T^{-1}\| \leqq \Big\|\sum_{i=1}^m \theta_i f_{i,m}\Big\|_F = \gamma_m^{-1}\Big\|T\sum_{i=1}^m \theta_i\phi_{i,m}\Big\|_F \leqq \gamma_m^{-1}\|T\|. \tag{4.11}$$

As shown in [1, p. 470] there is a constant K depending only on the function F (and thus independent of m) such that

$$E\Big(\int_0^1 F\Big(\Big|\sum_{i=1}^m \theta_i f_{i,m}(t)\Big|\Big)\,dt\Big) \leqq K\sum_{i=1}^m \int_0^1 F(|f_{i,m}(t)|)dt,$$

where E denotes the average of all possible 2^m choices of $\theta = \{\theta_i\}$. In particular, for at least one-half of the 2^m choices of signs, we have

$$\int_0^1 F\Big(\Big|\sum_{i=1}^m \theta_i f_{i,m}(t)\Big|\Big)dt \leqq 2K\sum_{i=1}^m \int_0^1 F(|f_{i,m}(t)|)dt. \tag{4.12}$$

A simple combinatorial argument shows therefore that we can choose inductively the signs $\{\theta_i^m\}_{i=1}^m$ for $m = 2^n$, $n = 1, 2,\cdots$ so that (4.12) holds with $\theta_i = \theta_i^m$ and so that the functions

$$\psi_n = \sum_{i=1}^m \theta_i^m\phi_{i,m} \qquad (m = 2^n) \qquad n = 1,2,\ldots$$

are asymptotically orthogonal in the sense that for every k there is an $n(k)$ such that

$$\Big|\int_0^1 \psi_j(t)\psi_n(t)dt\Big| < 2^{-k}, \qquad 1 \leqq j \leqq k, \qquad n \geqq n(k). \tag{4.13}$$

It follows from (4.13), the fact that $|\psi_n(t)| \equiv 1$, and Khintchine's inequality that for every sequence n_j tending sufficiently fast to ∞, the span of $\{\psi_{n_j}\}$ in L_G is isomorphic to l_2.

Put $h_n = \gamma_{2^n}^{-1} T\psi_n \in L_F$, $n=1,2,\cdots$. Notice that there is an $\varepsilon > 0$ such that $h_n \in A_\varepsilon^F$ for every n. Indeed, otherwise it would follow from Proposition 3 that for some sequence $\{n_j\}$, the functions $\gamma_{2^{n_j}} h_{n_j}$: $j = 1,2,\cdots$ are equivalent to the unit vector basis of l_H for some $H \in C_F^\infty$. By the remark in the preceding paragraph and the fact that T is an isomorphism, it follows that l_H is isomorphic to l_2. However, since $\beta_F^\infty < 2$ the set C_F^∞ contains no function equivalent to x^2 and we arrive at a contradiction.

Since $h_n \in A_\varepsilon^F$, $n = 1,2,\cdots$ we have

$$\varepsilon F(\varepsilon \| h_n \|_F) \leqq \int_0^1 F(|h_n(t)|)dt \qquad n = 1,2,\cdots.$$

By (4.10) and (4.12) it follows that

$$\varepsilon F(\varepsilon \| h_n \|_F) \leqq 2CK2^n$$

and thus by (4.11)

$$\varepsilon F(\varepsilon / \gamma_{2^n} \| T^{-1} \|) \leqq 2CK2^n.$$

Since F satisfies the Δ_2 condition it follows that for some constant K_1 independent of n, $F(\gamma_{2^n}^{-1}) \leqq K_1 2^n$. Thus if $2^n \leqq G(x) \leqq 2^{n+1}$ then $x \leqq \gamma_{2^{n+1}}^{-1}$ and hence

$$F(x) \leqq K_1 2^{n+1} \leqq 2K_1 G(x)$$

which concludes the proof of the theorem.

COROLLARY. *Let F and G be two reflexive Orlicz functions such that $2 \notin [\alpha_F^\infty, \beta_F^\infty]$. If L_F is isomorphic to L_G then F is equivalent to G at ∞.*

PROOF. Assume first that $\beta_F^\infty < 2$. By passing to the dual and using Proposition 4 and its Corollary, it follows that $\beta_G^\infty = \beta_F^\infty$. Hence by using twice Theorem 4 it follows that F is equivalent to G. The case $\alpha_F^\infty > 2$ is obtained from the previous one by duality.

REMARKS. We do not know whether the Corollary is true without the assumption $2 \notin [\alpha_F^\infty, \beta_F^\infty]$. The theorem clearly fails without assuming $\beta_F^\infty < 2$ since L_2 is isomorphic to a subspace of L_p for every p.

REFERENCES

1. J. Bretagnolle and D. Dacunha-Castelle, *Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces L^p*, Ann. Ec. Norm. Supér. **2** (1969), 437–480.

2. V. F. Gaposhkin, *Existence of absolute bases in Orlicz spaces*, Funkcional. Anal. Priložen. **1** (1967), 26–32.

3. M. I. Kadec and A. Pelczynski, *Bases lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1962), 161–176.

4. M. A. Krasnoselskii and Ya. B. Rutickii, *Convex functions and Orlicz spaces*, Groningen (The Netherlands), 1961.

5. K. J. Lindberg, *Contractive projections in Orlicz sequence spaces and continuous function spaces*, Ph.D. thesis, University of California, Berkeley, 1970.

6. J. Lindenstrauss, *A remark on symmetric bases*, Israel J. Math. **13** (1972), 317-320.

7. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications*, Studia Math. **29** (1968), 275-326.

8. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.

9. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces II*, Israel J. Math. **11** (1972), 355–379.

10. W. Matuszewska and W. Orlicz, *On certain properties of ϕ-functions*, Bull. Acad. Polon. Sci. **8** (1960), 439–443.

11. V. Milman, *The geometric theory of Banach spaces, Part II*, Uspehi Mat. Nauk. **26** (1971), 73–149.

12. H. R. Pitt, *A note on bilinear forms*, J. London Math. Soc. **11** (1936), 174–180.

13. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 2nd edition, Springer-Verlag, 1954.

14. E. M. Semenov, *A new interpolation theorem*, Funkkcional. Anal. Priložen. 2 (1968), 68–80.

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